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ON ONE FAMILY OF MINIMAL TORI IN $\mathbb{R}^3$ WITH PLANAR EMBEDDED ENDS

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Abstract: We construct new examples of complete minimal tori in the three-dimensional Euclidean space with an arbitrary even number $n \geq 6$ of planar embedded ends.

Keywords: minimal surface, Willmore surface, superminimal surface

§ 1. Introduction

In this article we construct complete minimal immersions of tori in $\mathbb{R}^3$ punctured at arbitrary even number $n \geq 6$ of points. Asymptotically, in a neighborhood of punctured points a torus looks like a plane. A surface with such behavior in a neighborhood of punctured points is called a *surface with planar embedded ends*.

The main result of this paper is the following

Theorem 1. *For every even $n \geq 6$, there is a complete minimal immersion of a torus in $\mathbb{R}^3$ with n planar embedded ends.*

A priori constructed tori can have branch points at which the induced metric degenerates. By now, it is proven that for a small number of ends ($n = 6, 8, 10$) there are tori without branch points [1]; however, this is most likely to be true for an arbitrary even n.

The study of complete minimal surfaces with planar embedded ends was initiated by Bryant. In [2] he demonstrated that an inversion takes complete minimal surfaces with planar embedded ends into Willmore surfaces, i.e., into extremals of the Willmore functional

$$W(\Sigma) = \int_{\Sigma} \left(H^2 - K\right) d\sigma,$$

where H and K are the mean and Gaussian curvatures and $d\sigma$ is the area element of a surface Σ. In this event, the planar ends go into a multiple point of the surface and the value of the Willmore functional is equal to $4\pi n$, where n is the multiplicity of the point (or the number of planar ends). Bryant demonstrated that all Willmore spheres are obtained in this way.

A minimal sphere with one planar end ($n = 1$) is the standard plane. In [3] Bryant proved that there are no minimal spheres with planar ends for $n = 2, 3, 5, 7$. In [4] Peng constructed some examples of minimal spheres with n planar ends for even $n \geq 4$ and odd $n \geq 9$.

In the case of tori, the images of minimal surfaces with planar ends determine the class of superminimal Willmore tori, while the other Willmore tori are described by means of solutions to 4-particle Toda lattices (for example, see [5]).

For obvious reasons, there are no minimal tori for $n = 1, 2$. Kusner and Schmitt [6] demonstrated that there are no complete minimal tori with three planar ends. A complete minimal torus with four planar ends in $\mathbb{R}^3$ was constructed by Costa [7] and by Kusner and Schmitt [6]. We constructed an example of a torus with six planar ends [1].

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We now describe our construction briefly. Consider the Riemann surface Γ of genus 1 given in $\mathbb{C}^2$ by the equation

$$w^2 = P(z) = 4(z-p_1)(z-p_2)(z-p_3), \quad p_1, p_2, p_3 \in \mathbb{R}. \tag{1}$$

Let γ_1 and γ_2 be generators of $\pi_1(\Gamma)$.

We use the Weierstrass representation [8] to define the minimal immersions $\Phi : \Gamma \to \mathbb{R}^3$

$$\Phi(T) = \mathrm{Re} \int_{T_0}^{T} \big(\psi_1^2 - \psi_2^2, i\big(\psi_1^2 + \psi_2^2\big), 2\psi_1\psi_2\big) \frac{dz}{w}, \tag{2}$$

where $T_0 \in \Gamma$ is a fixed point and ψ_1^2, ψ_2^2, and $\psi_1\psi_2$ are meromorphic functions on Γ. Consequently, the following equalities are valid on the universal covering $\upsilon : \Upsilon \to \Gamma$ for $\gamma : [0,1] \to \Upsilon$ such that $\upsilon(\gamma)$ is homotopically equivalent to γ_1 or γ_2:

$$\psi_1(\gamma(0)) = \varepsilon(\gamma)\psi_1(\gamma(1)), \quad \psi_2(\gamma(0)) = \varepsilon(\gamma)\psi_2(\gamma(1)), \quad \varepsilon(\gamma) = \pm 1.$$

Then ψ_1 and ψ_2 are sections of the spin structure on Γ. In our construction $\varepsilon(\gamma_1)$ and $\varepsilon(\gamma_2)$ are equal to 1; i.e., ψ_1 and ψ_2 are meromorphic functions and (2) cannot be an embedding [6].

A mapping Φ is a minimal immersion with planar ends if and only if each pole of the differentials

$$\psi_1^2 \frac{dz}{w}, \quad \psi_2^2 \frac{dz}{w}, \quad \psi_1\psi_2 \frac{dz}{w}$$

is of *second* order with *zero residues* [6].

Moreover, for Φ to be correctly defined, we have to solve the period problem:

$$\mathrm{Re} \int_{\gamma_i} \big(\psi_1^2 - \psi_2^2\big) \frac{dz}{w} = 0, \quad \mathrm{Im} \int_{\gamma_i} \big(\psi_1^2 + \psi_2^2\big) \frac{dz}{w} = 0, \quad \mathrm{Re} \int_{\gamma_i} 2\psi_1\psi_2 \frac{dz}{w} = 0, \tag{3}$$

where $i = 1, 2$.

In our construction the functions ψ_1 and ψ_2 are as follows:

$$\psi_1 = \sum_{i=1}^{m} \frac{\alpha_i w}{z - p_i}, \quad \psi_2 = \sum_{j=1}^{m} \frac{\beta_j w}{z - p_j}, \tag{4}$$

where $m \geq 4$ is an integer, $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m \in \mathbb{C}$, and $p_1, \dots, p_m \in \mathbb{R}$ are distinct. It is easy to verify that $\psi_1 \frac{dz}{w}$ and $\psi_2 \frac{dz}{w}$ have poles of the first order at the branch points

$$P_0 = (\infty, \infty), \quad P_1 = (p_1, 0), \quad P_2 = (p_2, 0), \quad P_3 = (p_3, 0)$$

and at the points

$$P_j^- = (p_j, -\sqrt{P(p_j)}), \quad P_j^+ = (p_j, \sqrt{P(p_j)}), \quad j = 4, \dots, m.$$

Define the space $V(p) = V(p_1, \dots, p_m)$ of functions of the form (4)

$$V(p) = \left\{ \sum_{i=1}^{m} \frac{\nu_i w}{z - p_i} \mid (\nu_1, \dots, \nu_m) \in \ker M \subset \mathbb{C}^m \right\},$$

where the matrix

$$M = \begin{pmatrix} \frac{1}{p_4 - p_1} & \frac{1}{p_4 - p_2} & \frac{1}{p_4 - p_3} & \frac{P'(p_4)}{4P(p_4)} & \frac{1}{p_4 - p_5} & \cdots & \frac{1}{p_4 - p_m} \\ \frac{1}{p_5 - p_1} & \frac{1}{p_5 - p_2} & \frac{1}{p_5 - p_3} & \frac{1}{p_5 - p_4} & \frac{P'(p_5)}{4P(p_5)} & \cdots & \frac{1}{p_5 - p_m} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{p_m - p_1} & \frac{1}{p_m - p_2} & \frac{1}{p_m - p_3} & \frac{1}{p_m - p_4} & \frac{1}{p_m - p_5} & \cdots & \frac{P'(p_m)}{4P(p_m)} \end{pmatrix}$$

acts on a column vector by left multiplication.

Proposition 1. 1. *For every $\psi_1, \psi_2 \in V(p)$ we have*

$$\operatorname*{res}_{Q} \frac{\psi_1\psi_2 dz}{w} = 0,$$

where $Q = P_0, \dots, P_3, P_4^{\pm}, \dots, P_m^{\pm}$.

2. *The following equality is valid for almost all $p_1, \dots, p_m \in \mathbb{R}$ such that $p_1 + p_2 + p_3 = 0$:*

$$\dim_{\mathbb{C}} V(p) = 3.$$

Thus, the Weierstrass representation (2) for $\psi_1, \psi_2 \in V(p)$ determines a minimal surface with planar ends.

We now have to choose ψ_1 and ψ_2 in $V(p)$ to solve the six real equations (3). Here we have six free complex parameters (since ψ_1, $\psi_2 \in V(p)$ and $\dim_{\mathbb{C}} V(p) = 3$ by Proposition 1).

The following proposition plays the key role in solving the period problem (3):

Proposition 2. *There are symmetric bilinear forms*

$$A : V(p) \times V(p) \to \mathbb{C}, \quad B(p) : V(p) \times V(p) \to \mathbb{C}$$

such that

$$-\eta_k A\big(\psi_1, \psi_2\big) + \omega_k B(p; \psi_1, \psi_2) = \frac{1}{8} \int\limits_{\gamma_k} \psi_1\psi_2 \frac{dz}{w}, \quad k = 1, 2, \tag{5}$$

where

$$\eta_k = -\frac{1}{2} \int\limits_{\gamma_k} \frac{z dz}{w}, \quad \omega_k = \frac{1}{2} \int\limits_{\gamma_k} \frac{dz}{w}, \quad k = 1, 2.$$

The form A is independent of $p_1, \dots, p_m$ and positive definite on $V(p)$.

By the positive definiteness of A we mean the following:

$$A(\psi, \psi) > 0 \text{ for } \psi = \sum_{i=1}^{m} \frac{\nu_i w}{z - p_i} \in V(p), \quad \big(\nu_1, \dots, \nu_m\big) \in \mathbb{R}^n \backslash \{0\}.$$

The positive definiteness of A implies that we can choose a basis ξ_1, ξ_2, ξ_3 for the space $V(p)$ such that A is given by the identity matrix and $B(p)$ is diagonal. Thus, we have

Lemma 1. *There is a basis ξ_1, ξ_2, ξ_3 for the space $V(p)$ such that*

$$\int\limits_{\gamma} \xi_i \xi_j \frac{dz}{w} = 0, \quad i, j = 1, \dots, 3,$$

for $i \neq j$ and $\gamma = \gamma_1, \gamma_2$.

Existence of such a basis enables us to solve explicitly the period problem (3).

Let

$$(p_1, \dots, p_m) = \begin{cases} (-1, 0, 1, 2, 2 + t, 3, \dots, \frac{m-1}{2} + t) & \text{for odd } m, \\ (-1, 0, 1, 2, 2 + t, 3, \dots, \frac{m}{2}, 1 + t) & \text{for even } m \end{cases}$$

for a sufficiently small $t \in \mathbb{R}$.

Proposition 3. *Put* $\psi_1 = v(r,s)\xi_1 + \xi_2$, $\psi_2 = x(r,s)\xi_1 + y(r,s)\xi_2 + u(r,s)\xi_3$, *where*

$$v(r,s) = \pm\sqrt{-\frac{1}{2c}(|c|^2 + 2i\,\mathrm{Im}\,d \pm \sqrt{(|c|^2 + 2i\,\mathrm{Im}\,d)^2 - 4|c|^2 d})}; \tag{6}$$

$$x(r,s) = \frac{-a_2 r - b_2 s}{v(r,s)}; \quad y(r,s) = a_1 r + b_1 s;$$

$$u(r,s) = \sqrt{\frac{1}{a_3}(a_1 \overline{v(r,s)}^2 - a_1 x^2(r,s) - a_2 y^2(r,s) + a_2)}; \tag{7}$$

$$a_k = \int\limits_{\gamma_1} \xi_k^2 \frac{dz}{w}, \quad b_k = \int\limits_{\gamma_2} \xi_k^2 \frac{dz}{w}, \quad k = 1,2,3; \tag{8}$$

$$c(r,s) = \frac{a_2 b_3 + a_3 b_2 - (a_2 b_3 - a_3 b_2)(a_1 r + b_1 s)^2}{(a_1 b_3 + a_3 b_1)};$$

$$d(r,s) = -\frac{(a_1 b_3 - a_3 b_1)}{(a_1 b_3 + a_3 b_1)}(a_2 r + b_2 s)^2.$$

Then there is an open domain $\Omega \subset \mathbb{R}^2$ *such that* (3) *holds for almost all* $(r,s) \in \Omega$.

Theorem 1 follows from Propositions 1–3 and 5.

Thus, we obtain a two-parameter family of tori with planar ends for fixed $p_1, \dots, p_m$.

Immersion (2) has no branch points if $(|\psi_1|^2 + |\psi_2|^2)(T) \neq 0$ for all $T \in \Gamma$. This condition depends on $m+2$ free parameters $r, s, p_1, \dots, p_m$ which makes the inequality obvious for parameters in general position; however, a rigorous proof of this assertion is technically complicated. As mentioned above, we can show this only for small $n = 2m - 2$.

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§ 2. Proof of Theorem 1

Prove some auxiliary lemmas.

Lemma 2. *For every function* $\psi \in V(p)$ *we have*

$$\operatorname*{res}_Q \frac{\psi^2 dz}{w} = 0, \quad Q = P_0, \dots, P_3, P_4^{\pm}, \dots, P_m^{\pm}.$$

The dimension of the space $V(p)$ *over* $\mathbb{C}$ *is greater than or equal to* 3.

Proof. Consider an arbitrary function in $V(p)$:

$$\psi = \sum_{i=1}^{m} \frac{\nu_i w}{z - p_i}.$$

The holomorphic involution $\sigma : (z,w) \mapsto (z,-w)$ exists on the torus Γ. The vanishing of the residues $\psi^2 dz/w$ at the branch points of Γ follows from the obvious equality

$$\sigma^*(\psi^2 dz/w) = -\psi^2 dz/w$$

and the invariance of the points $P_0, \dots, P_3$ under the involution σ.

Choose a local parameter $q = z - p_k$ in a neighborhood of the point P_k^+, $k = 4, \dots, m$. Denote $\sqrt{P(p_k)}$ by w_k and $\frac{dw}{dq}(p_k, w_k)$ by w_k' for $k = 4, \dots, m$. The Laurent series expansion of the differential $\frac{\psi^2}{w} dq$ in a neighborhood of P_k^+ has the form

$$\frac{\psi^2}{w} dq = \frac{\nu_k^2 w_k}{q^2}\, dq + \left(\frac{q^2 \psi^2}{w}\right)' (P_k^+) \frac{1}{q}\, dq + O(1)\, dq.$$

Consequently, the residue at P_k^+ is equal to

$$\operatorname*{res}_{P_k^+} \frac{\psi^2}{w} dq = \lim_{q \to 0} \frac{1}{w^2}((q^2\psi^2)'w - q^2\psi^2 w'). \tag{9}$$

It is obvious that

$$\lim_{q \to 0} q^2\psi^2 w' = \nu_k^2 w_k^2 w_k'. \tag{10}$$

Using the equality $(q^2\psi^2)'w = 2(q\psi)'q\psi w$, we calculate $(q\psi)'$:

$$\begin{aligned}(q\psi)'\big(P_k^+\big) &= \Bigg(w\frac{\nu_k}{z-p_k}(z-p_k) + w\sum_{i=1, i\neq k}^{m} \frac{\nu_i}{z-p_i}(z-p_k)\Bigg)\big(P_k^+\big) \\ &= \nu_k w_k' + w_k \sum_{i=1, i\neq k}^{m} \frac{\nu_i}{p_k - p_i}. \end{aligned} \tag{11}$$

Inserting (10) and (11) in (9), we obtain

$$\begin{aligned}\operatorname*{res}_{P_k^+} \frac{\psi^2}{w}\, dq &= \frac{1}{w_k^2}\Bigg(2\Bigg(\nu_k w_k' + w_k \sum_{i=1, i\neq k}^{m} \frac{\nu_i}{p_k-p_i}\Bigg)\nu_k w_k^2 - \nu_k^2 w_k^2 w_k'\Bigg) \\ &= 2\nu_k w_k \Bigg(\frac{P'(p_k)}{4P(p_k)}\nu_k + \sum_{i=1, i\neq k}^{m} \frac{1}{p_k - p_i}\nu_i\Bigg). \end{aligned} \tag{12}$$

The last equality is valid, since on the torus $w^2 = P(z)$ we have

$$\frac{w_k'}{2w_k} = \frac{P'(p_k)}{4P(p_k)}, \quad k = 4, \dots, m.$$

Since the condition $(\nu_1, \dots, \nu_m) \in \ker M$ implies the equalities

$$\frac{P'(p_k)}{4P(p_k)}\nu_k + \sum_{i=1, i\neq k}^{m} \frac{1}{p_k - p_i}\nu_i = 0, \quad k = 4, \dots, m,$$

it follows from (12) that $\operatorname{res}_{P_k^\pm} \frac{\psi^2 dz}{w} = 0$ for $\psi \in V(p)$, $k = 4, \dots, m$.

Thus, the 1-form under study have zero residues at each of the points $P_0, \dots, P_3, P_4^\pm, \dots, P_m^\pm$.

The rank of M does not exceed $m-3$. The dimension of $V(p)$ is equal to $\dim_{\mathbb{C}} \ker M = m - \operatorname{rank} M \geq 3$. Lemma 2 is proven.

Lemma 3. *For almost all points $p_1, \dots, p_m \in \mathbb{R}$ such that $p_1 + p_2 + p_3 = 0$ the matrix*

$$M_{4,\dots,m} = \begin{pmatrix} \frac{P'(p_4)}{4P(p_4)} & \frac{1}{p_4 - p_5} & \dots & \frac{1}{p_4 - p_m} \\ \frac{1}{p_5 - p_4} & \frac{P'(p_5)}{4P(p_5)} & \dots & \frac{1}{p_5 - p_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{p_m - p_4} & \frac{1}{p_m - p_5} & \dots & \frac{P'(p_m)}{4P(p_m)} \end{pmatrix}$$

is nondegenerate.

Proof. Let $t \in \mathbb{R}$, $\varepsilon = O(t)$, and

$$(p_1, \dots, p_m) = \begin{cases} \big(-1, 0, 1, 2, 2+t, 3, \dots, \frac{m-1}{2} + t\big) & \text{for odd } m, \\ \big(-1, 0, 1, 2, 2+t, 3, \dots, \frac{m}{2}, 1+t\big) & \text{for even } m. \end{cases}$$

Then for the determinant we have the estimate

$$t\det M_{4,\dots,m}=\begin{vmatrix}\varepsilon & -1 & \dots & \varepsilon & \varepsilon\\ 1 & \varepsilon & \dots & \varepsilon & \varepsilon\\ \vdots & \vdots & \ddots & & \vdots\\ \varepsilon & \varepsilon & \dots & \varepsilon & -1\\ \varepsilon & \varepsilon & \dots & 1 & \varepsilon\end{vmatrix}=1+O(\varepsilon)$$

in the case of an odd m and the estimate

$$t\det M_{4,\dots,m}=\begin{vmatrix}\varepsilon & -1 & \dots & \varepsilon & \varepsilon & \varepsilon\\ 1 & \varepsilon & \dots & \varepsilon & \varepsilon & \varepsilon\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ \varepsilon & \varepsilon & \dots & \varepsilon & -1 & \varepsilon\\ \varepsilon & \varepsilon & \dots & 1 & \varepsilon & \varepsilon\\ \varepsilon & \varepsilon & \dots & \varepsilon & \varepsilon & \frac{11}{24}\end{vmatrix}=\frac{11}{24}+O(\varepsilon)$$

in the case of an even m as $t\to 0$. Consequently, the determinant $M_{4,\dots,m}$ is a nonzero rational function of $p_1,\dots,p_m$. Since the zero set of a rational function has measure zero, the determinant $\det M_{4,\dots,m}$ is different from zero for almost all $p_1,\dots,p_m\in\mathbb{R}$. Lemma 3 is proven.

Proof of Proposition 1.

1. Take arbitrary $\psi_1,\psi_2\in V(p)$. Then $\psi_1-\psi_2$ and $\psi_1+\psi_2$ belong to $V(p)$. By Lemma 1, for the residues at the points $P_0,\dots,P_3,P_4^{\pm},\dots,P_m^{\pm}$ we have

$$\operatorname{res}(\psi_1+\psi_2)^2\frac{dz}{w}=0,\quad \operatorname{res}(\psi_1-\psi_2)^2\frac{dz}{w}=0,$$

$$\operatorname{res}\frac{\psi_1\psi_2dz}{w}=\frac{1}{4}\operatorname{res}\left((\psi_1+\psi_2)^2-(\psi_1-\psi_2)^2\right)\frac{dz}{w}=0.$$

2. By Lemma 2, for almost all points $p_1,\dots,p_m\in\mathbb{R}$ such that $p_1+p_2+p_3=0$ the rank of M is equal to $m-3$. Consequently, the dimension of $V(p)$ is equal to $\dim_{\mathbb{C}}\ker M=m-\operatorname{rank}M=3$.

The proposition is proven.

Proof of Proposition 2. Henceforth we assume that $p_1+p_2+p_3=0$.

Represent Γ by gluing two copies of the planes $\overline{\mathbb{C}}$ ("lower" and "upper" sheets) with cuts along the intervals $[p_1,p_2]$ and $[p_3,\infty]$. In this representation to the points (w,z) and $(-w,z)$ in Γ there correspond the points z on the "lower" and "upper" sheets of the Riemann surface. Let

$$\gamma_1=\{(z,w)\in\Gamma\mid z\in[p_1,p_2]\},\quad \gamma_2=\{(z,w)\in\Gamma\mid z\in[p_2,p_3]\}.$$

These cycles γ_1 and γ_2 are homotopically equivalent to nontrivial cycles of a torus demonstrated in Fig. 1.

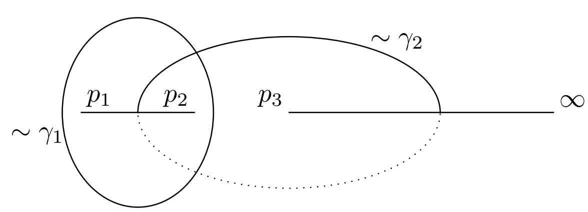

Fig. 1

The doted line stands for the part of the cycle lying on the "lower" sheet of the Riemann surface and the solid line stands for the part of the cycle on the "upper" sheet. The cycles γ_1 and γ_2 constitute a basis for $H_1(\Gamma;\mathbb{Z}_2)$.

Define the sequence $\delta_i=\begin{cases}1 & \text{for } i=1,2,3,\\ 1/2 & \text{for } i\geq 4.\end{cases}$

Lemma 4. *The following equalities are valid for ψ_1, $\psi_2 \in V(p)$:*

$$\int\limits_{\gamma_k} \frac{\psi_1\psi_2}{w}\,dz = -8\biggl(\sum_{j=1}^{m}\sum_{i=1}^{m}\alpha_i\beta_j + \sum_{i=1}^{m}\delta_i\alpha_i\beta_i\biggr)\eta_k$$
$$+8\biggl(\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\beta_j(p_i+p_j) - \sum_{i=1}^{m}\delta_i\alpha_i\beta_i p_i\biggr)\omega_k, \quad k=1,2,$$

where

$$\eta_k = -\frac{1}{2}\int\limits_{\gamma_k}\frac{z\,dz}{w}, \quad \omega_k = \frac{1}{2}\int\limits_{\gamma_k}\frac{dz}{w}, \quad k=1,2.$$

Proof. Consider the torus $\mathbb{T} = \mathbb{C}/\{2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}\}$ with a local parameter u. On $\mathbb{T}$ there is a unique meromorphic function with second-order pole at 0 having the following expansion about zero:

$$\wp(u) = \frac{1}{u^2} + o(u) + \dots.$$

The function $\wp(u)$ is called the *Weierstrass $\wp$ function* [9].

For $p_1 + p_2 + p_3 = 0$ the mapping $\rho(u) : \mathbb{T} \to \Gamma$ given by the formula $\rho(u) = (\wp(u), \wp'(u))$ is biholomorphic [9]. Therefore, the equality

$$(\wp'(u))^2 = 4(\wp(u)-p_1)(\wp(u)-p_2)(\wp(u)-p_3) \tag{13}$$

is valid and ρ maps the points 0, ω_1, ω_2, and $\omega_3 = \omega_1 + \omega_2$ onto $P_0, \dots, P_3$ in some order.

The cycles $2\omega_1 t$ and $2\omega_2 t$, $t \in [0,1]$, constitute a basis for $H_1(\mathbb{T};\mathbb{Z})$. Thereby the images $\rho(2\omega_1 t)$ and $\rho(2\omega_2 t)$ of these cycles are homotopically equivalent to γ_1 and γ_2 respectively [9].

Choose $u_1, \dots, u_m$ such that

$$\rho(u_i) = (p_i, w_i), \quad i = 1, \dots, m.$$

It follows from (13) then that $\rho(-u_i) = (p_i, -w_i)$, $i = 4, \dots, m$. Assume that $w_1 = w_2 = w_3 = 0$.

By Proposition 1, the differential $\psi_1\psi_2 dz/w$ has only second-order poles without residues; therefore, $\psi_1\psi_2 dz/w$ is a linear combination of the differentials du, $\wp(u)du$, $\wp(u-u_1)du, \dots, \wp(u-u_m)du$, $\wp(u+u_4)du, \dots, \wp(u+u_m)du$.

Find this linear combination. Note that the equality $\rho^*(dz/w) = du$ is valid. Let

$$\alpha_0 = \operatorname*{res}_{Q}\frac{\psi_1}{w}\,dz, \quad \beta_0 = \operatorname*{res}_{Q}\frac{\psi_2}{w}\,dz$$

at a point $Q \in \{P_0, \dots, P_3, P_4^{\pm}, \dots, P_m^{\pm}\}$. Take $u_0 \in \mathbb{T}$ such that $\rho(u_0) = Q$. Then, by Proposition 1,

$$\rho^*\frac{\psi_1\psi_2}{w}\,dz = \frac{\alpha_0\beta_0}{(u-u_0)^2}du + O(1)\,du.$$

The form of ψ_1 implies that for the residues of $\psi_1 dz/w$ we have

$$\operatorname*{res}_{P_0}\frac{\psi_1}{w}\,dz = -2\sum_{i=1}^{m}\alpha_i, \ \operatorname*{res}_{P_1}\frac{\psi_1}{w}\,dz = 2\alpha_1, \dots, \operatorname*{res}_{P_3}\frac{\psi_1}{w}dz = 2\alpha_3,$$
$$\operatorname*{res}_{P_4^{\pm}}\frac{\psi_1}{w}\,dz = \alpha_4, \dots, \operatorname*{res}_{P_m^{\pm}}\frac{\psi_1}{w}\,dz = \alpha_m.$$

The residues of $\psi_2 dz/w$ are calculated similarly.

Thus, the sought linear combination $\rho^* \frac{\psi_1\psi_2}{w}\, dz$ is equal to

$$\left(4\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\beta_j\wp(u) + 4\sum_{i=1}^{3}\alpha_i\beta_i\wp(u-u_i) + \sum_{i=4}^{m}\alpha_i\beta_i(\wp(u-u_i)+\wp(u+u_i)) + c_0\right) du.$$

Define the constant c_0 from the behavior of the 1-forms near $\rho(0)=\infty$.

The so-found linear combination for $\rho^* \frac{\psi_1\psi_2}{w}\, dz$ has the following expansion about zero:

$$4\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\beta_j\frac{du}{u^2} + \left(4\sum_{i=1}^{m}\delta_i\alpha_i\beta_i\wp(u_i) + c_0\right) du + O(u)\, du. \tag{14}$$

Choose the local parameter $q = \frac{1}{\sqrt{z}}$. Then in a neighborhood of ∞ we have

$$(z,w) = \left(\frac{1}{q^2}, \frac{2}{q^3}\sqrt{(1-p_1q^2)(1-p_2q^2)(1-p_3q^2)}\right)$$

and

$$du = \frac{dz}{w} = -(1+O(q))\, dq.$$

To write down the Laurent series of the differential $\psi_1\psi_2 dz/w$ at the point ∞, we use the asymptotic expansion

$$\sum_{i=1}^{m}\frac{\alpha_i w}{z-p_i} = q^2\sum_{i=1}^{m}\alpha_i(1+p_iq^2+O(q^4))$$

and

$$qw\, dq = \frac{1}{q^2}\sqrt{(1-p_1q^2)(1-p_2q^2)(1-p_3q^2)} = \frac{1}{q^2}(1-(p_1+p_2+p_3)q^2+O(q^4)).$$

Since $p_1+p_2+p_3=0$, the last expression is equal to $\frac{1}{q^2}+O(q^2)$.

Write down the expansion of $\psi_1\psi_2 dz/w$ as follows:

$$\begin{aligned}
\sum_{i=1}^{m}\frac{\alpha_i w}{z-p_i}\sum_{j=1}^{m}\frac{\beta_j w}{z-p_j}\frac{dz}{w} &= -4\sum_{i=1}^{m}\frac{\alpha_i}{1-p_iq^2}\sum_{j=1}^{m}\frac{\beta_j}{1-p_jq^2}\, qwdq \\
= -4\left(\sum_{i,j=1}^{m}\alpha_i\beta_j\frac{1}{q^2} + \sum_{i=1}^{m}\alpha_i(1+p_iq^2+O(q^4))\sum_{j=1}^{m}\beta_j(1+p_jq^2+O(q^4))\right) dq& \\
+O(q)\, dq = -4\sum_{i,j=1}^{m}\alpha_i\beta_j\frac{dq}{q^2} - 4\sum_{i,j=1}^{m}\alpha_i\beta_j(p_i+p_j)dq + O(q)dq.&
\end{aligned} \tag{15}$$

Equating the expansions (14) and (15) of the differential $\psi_1\psi_2 dz/w$, we find c_0:

$$c_0 = 4\sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i\beta_j(p_i+p_j) - 4\sum_{i=1}^{m}\delta_i\alpha_i\beta_ip_i.$$

From the relations

$$\int_{\gamma_k}\wp(u-u_i)\, du = \int_{\gamma_k}\wp(u)\, du = \int_{\gamma_k}\frac{zdz}{w} = -2\eta_k, \quad \int_{\gamma_k} du = 2\omega_k,$$

for $i=1,\dots,m$ and $k=1,2$, we obtain the assertion of the lemma.

Continue the proof of Proposition 2. It follows from Lemma 4 that A is independent of the choice of $p_1, \dots, p_m$. We denote the quadratic form corresponding to the symmetric bilinear forms A and $B(p)$ by A and $B(p)$. Write down these matrices:

$$A = \begin{pmatrix} 1+\delta_1 & 1 & \dots & 1 \\ 1 & 1+\delta_2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1+\delta_m \end{pmatrix}, \qquad B(p) = \begin{pmatrix} (2-\delta_1)p_1 & p_2+p_1 & \dots & p_m+p_1 \\ p_1+p_2 & (2-\delta_2)p_2 & \dots & p_m+p_2 \\ \vdots & \vdots & \ddots & \vdots \\ p_1+p_m & p_2+p_m & \dots & (2-\delta_m)p_m \end{pmatrix}.$$

Recall that the *corner minor* of order k of a matrix is the determinant of the submatrix constituted by k upper rows and k left columns of the matrix.

Let $m \geq 3$ be an arbitrary number. The corner minors of orders $k = 1, 2, 3$ of the matrix A are equal to 2, 3, 4 respectively; i.e., they are positive. For each $k \geq 4$ the corner minor is equal to $2^{4-k}(k-1) > 0$. This can be demonstrated by elementary transformations of the matrix. Starting with the bottom row, from each row we subtract the previous. We stop with terminate this process at the second row:

$$\begin{pmatrix} 1+\delta_1 & 1 & \dots & 1 \\ 1 & 1+\delta_2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1+\delta_k \end{pmatrix} \sim \begin{pmatrix} 1+\delta_1 & 1 & \dots & 1 & 1 & 1 \\ -\delta_1 & \delta_2 & \dots & 0 & 0 & 0 \\ 0 & -\delta_2 & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\delta_{k-2} & \delta_{k-1} & 0 \\ 0 & 0 & \dots & 0 & -\delta_{k-1} & \delta_k \end{pmatrix}.$$

We now start a new series of transformations with the last but one column: to each column we add the previous. We stop with terminate the transformation of the matrix at the fourth column:

$$\begin{pmatrix} 1+\delta_1 & 1 & 1 & k-3 & k-2 & \dots & 3 & 2 & 1 \\ -\delta_1 & \delta_2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -\delta_2 & \delta_3 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -\delta_3 & \delta_4 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_5 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \delta_{k-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \delta_{k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \delta_k \end{pmatrix}.$$

Now, it suffices to add the doubled fourth column to the third column to make the submatrix split. The cofactor of the corner minor of order 3 is diagonal and its determinant is equal to $\frac{1}{2^{k-3}}$, while the determinant of the corner minor of order 3 is equal to $2(k-1)$. Consequently, the corner minor of order $k \geq 4$ is equal to $2^{4-k}(k-1)$.

In view of Sylvester's criterion, from positivity of all corner minors A we find that the quadratic form A is positive definite. The proposition is proven.

Write down the Gram–Schmidt matrix in the basis ξ_1, ξ_2, ξ_3:

$$(A(\xi_i, \xi_j))_{3\times 3} = (\delta_{ij}), \tag{16}$$

$$(B(\xi_i, \xi_j))_{3\times 3} = (\mu_i \delta_{ij}), \tag{17}$$

where $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$. For each $p = (p_1, \dots, p_m)$ the numbers μ_1, μ_2, and μ_3 are defined to within rearrangement.

Proposition 4. *There is a one-parameter family $\Gamma(p_t)$ of tori such that the periods*

$$\int\limits_{\gamma} \xi_1^2 \frac{dz}{w}, \quad \int\limits_{\gamma} \xi_2^2 \frac{dz}{w}, \quad \int\limits_{\gamma} \xi_3^2 \frac{dz}{w}$$

are pairwise distinct for $\gamma = \gamma_1, \gamma_2$.

Proof. Let

$$(p_1, \dots, p_m) = \begin{cases} \left(-1, 0, 1, 2, 2+t, 3, \dots, \frac{m-1}{2} + t\right) & \text{for odd } m, \\ \left(-1, 0, 1, 2, 2+t, 3, \dots, \frac{m}{2}, 1+t\right) & \text{for even } m, \end{cases}$$

where $t \in [0, 1)$.

Let

$$\zeta_k(t) = \sum_{i=1}^{3} \frac{\delta_{ik}}{z - p_i} + \sum_{j=4}^{m} \frac{\zeta_k^i(t)}{z - p_j(t)}, \quad k = 1, 2, 3,$$

where δ_{nk} is the Kronecker symbol and

$$\begin{pmatrix} \zeta_k^4(t) \\ \dots \\ \zeta_k^m(t) \end{pmatrix} = -M_{4,\dots,m}^{-1}(t) \begin{pmatrix} \frac{1}{p_4(t) - p_k} \\ \dots \\ \frac{1}{p_m(t) - p_k} \end{pmatrix}.$$

Since

$$M \begin{pmatrix} \delta_{1k} \\ \delta_{2k} \\ \delta_{3k} \\ \zeta_k^4(t) \\ \dots \\ \zeta_k^m(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{p_4(t) - p_k} \\ \dots \\ \frac{1}{p_m(t) - p_k} \end{pmatrix} + M_{4,\dots,m}(t) \begin{pmatrix} \zeta_k^4(t) \\ \dots \\ \zeta_k^m(t) \end{pmatrix},$$

the family of the functions $\zeta_1(t)$, $\zeta_2(t)$, and $\zeta_3(t)$ is a basis for $V(p_t)$.

Put

$$\zeta_k(0) = \begin{cases} \frac{1}{z - p_k} & \text{for odd } m, \\ \frac{1}{z - p_k} + \frac{4\delta_{3k}}{z - p_m} & \text{for even } m, \end{cases} \quad k = 1, 2, 3.$$

Lemma 5. *There is a sufficiently small $T > 0$ such that the functions $\zeta_k^i(t) : [0, T) \to \mathbb{R}$, $k = 1, 2, 3$, $i = 4, \dots, m$, are defined and continuous.*

Proof. Since

$$\frac{P'(z)}{P(z)} = \frac{1}{z - p_1} + \frac{1}{z - p_2} + \frac{1}{z - p_3}$$

and

$$p_i - p_j = \begin{cases} O(t) & \text{for } \{i, j\} = \{2k, 2k+1\} \text{ and some } k \geq 2, \\ O(1) & \text{otherwise}, \end{cases}$$

we have

$$tM_{4,\dots,m}(t) = \begin{pmatrix} \frac{P'(p_4)}{4P(p_4)} & \frac{1}{p_4 - p_5} & \dots & \frac{1}{p_4 - p_m} \\ \frac{1}{p_5 - p_4} & \frac{P'(p_5)}{4P(p_5)} & \dots & \frac{1}{p_5 - p_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{p_m - p_4} & \frac{1}{p_m - p_5} & \dots & \frac{P'(p_m)}{4P(p_m)} \end{pmatrix} \sim \begin{pmatrix} \varepsilon & -1 & \dots & \varepsilon & \varepsilon \\ 1 & \varepsilon & & \varepsilon & \varepsilon \\ \vdots & \vdots & \ddots & & \vdots \\ \varepsilon & \varepsilon & & \varepsilon & -1 \\ \varepsilon & \varepsilon & \dots & 1 & \varepsilon \end{pmatrix}$$

for odd m and

$$tM_{4,\dots,m}(t) \sim \begin{pmatrix} \varepsilon & -1 & \dots & \varepsilon & \varepsilon & \varepsilon \\ 1 & \varepsilon & & \varepsilon & \varepsilon & \varepsilon \\ \vdots & \vdots & \ddots & & & \vdots \\ \varepsilon & \varepsilon & & \varepsilon & -1 & \varepsilon \\ \varepsilon & \varepsilon & & 1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \dots & \varepsilon & \varepsilon & \frac{1}{4}+\varepsilon \end{pmatrix} \tag{18}$$

for even m, where $\varepsilon = O(t)$.

Now, it is obvious that, for a sufficiently small T, the quantity $|\det tM_{4,\dots,m}(t)|$ is greater than $c_1 = 1/8$ for all $t \in (0,T)$, and the absolute values of the entries of the matrix $tM_{4,\dots,m}(t)$ are less or equal to $c_2 = 1$. Since the entries of the inverse matrix are equal to the ratio of the cofactor and the determinant of the matrix, the inverse matrices $(tM_{4,\dots,m}(t))^{-1}$ exist and the absolute values of the entries of $(tM_{4,\dots,m}(t))^{-1}$ are less than $(m-1)!c_2^{m-1}/c_1 = 8(m-1)!$ for $t \in (0,T)$.

Continuity of $\zeta_1^i(t)$, $\zeta_2^i(t)$, $\zeta_3^i(t)$, $i = 4,\dots,m$, on $(0,T)$ is obvious. Show the right continuity at zero.

For an odd m the vectors

$$\begin{pmatrix} \frac{1}{p_4(t)-p_1} \\ \frac{1}{p_5(t)-p_1} \\ \dots \\ \frac{1}{p_{m-1}(t)-p_1} \\ \frac{1}{p_m(t)-p_1} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3+t} \\ \dots \\ \frac{2}{m+1} \\ \frac{2}{m+1+2t} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{p_4(t)-p_2} \\ \frac{1}{p_5(t)-p_2} \\ \dots \\ \frac{1}{p_{m-1}(t)-p_2} \\ \frac{1}{p_m(t)-p_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2+t} \\ \dots \\ \frac{2}{m-1} \\ \frac{2}{m-1+2t} \end{pmatrix}, \tag{19}$$

$$\begin{pmatrix} \frac{1}{p_4(t)-p_3} \\ \frac{1}{p_5(t)-p_3} \\ \dots \\ \frac{1}{p_{m-1}(t)-p_3} \\ \frac{1}{p_m(t)-p_3} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{1+t} \\ \dots \\ \frac{2}{m-3} \\ \frac{2}{m-3+2t} \end{pmatrix}$$

are bounded. Consequently, in this case the limit $\lim_{t\to+0} t(tM_{4,\dots,m}(t))^{-1}$ is equal to the zero matrix and the following equalities are valid:

$$\lim_{t\to+0} \zeta_1(t) = \zeta_1(0), \quad \lim_{t\to+0} \zeta_2(t) = \zeta_2(0), \quad \lim_{t\to+0} \zeta_3(t) = \zeta_3(0).$$

For even m, all coordinates but $\frac{1}{p_m(t)-p_3}$ of (19) are bounded. Consequently,

$$\lim_{t\to+0} \zeta_1(t) = \zeta_1(0), \quad \lim_{t\to+0} \zeta_2(t) = \zeta_2(0).$$

To find the limit $\lim_{t\to+0} \zeta_3(t)$, we eliminate the indeterminacy $0 \cdot \infty$ as follows:

$$-\lim_{t\to+0} t(tM_{4,\dots,m}(t))^{-1} \begin{pmatrix} \frac{1}{p_4(t)-p_3} \\ \frac{1}{p_5(t)-p_3} \\ \dots \\ \frac{1}{p_{m-1}(t)-p_3} \\ \frac{1}{p_m(t)-p_3} \end{pmatrix} = -\lim_{t\to+0} t\big(tM_{4,\dots,m}(t)\big)^{-1} \begin{pmatrix} 1 \\ \frac{1}{1+t} \\ \dots \\ \frac{2}{m-4} \\ 1/t \end{pmatrix}$$

$$= -\lim_{t\to+0} (tM_{4,\dots,m}(t))^{-1} \begin{pmatrix} 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}.$$

Denote $\lim_{t\to+0}(tM_{4,\dots,m}(t))^{-1}$ by

$$\frac{1}{\det tM_{4,\dots,m}(t)}\begin{pmatrix} \dots & \dots & \dots & C_1 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & C_{m-4} \\ \dots & \dots & \dots & C_{m-3} \end{pmatrix},$$

where C_j are the cofactors. Since all entries of the last row of the submatrix

$$C_j = (-1)^{j+m-3}\cdot\det\begin{pmatrix} \varepsilon & -1 & \dots & \varepsilon & \varepsilon \\ 1 & \varepsilon & \dots & \varepsilon & \varepsilon \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varepsilon & \varepsilon & \dots & \varepsilon & -1 \\ \varepsilon & \varepsilon & \dots & 1 & \varepsilon \\ \varepsilon & \varepsilon & \dots & \varepsilon & \varepsilon \end{pmatrix}$$

are equivalent to ε for $1 \le j \le m-4$, we have $\det C_j = \varepsilon$ as $t \to 0$.

The cofactor C_m is equivalent to

$$C_m = (-1)^{m-3+m-3}\cdot\det\begin{pmatrix} \varepsilon & -1 & \dots & \varepsilon & \varepsilon \\ 1 & \varepsilon & \dots & \varepsilon & \varepsilon \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \varepsilon & \varepsilon & \dots & \varepsilon & -1 \\ \varepsilon & \varepsilon & \dots & 1 & \varepsilon \end{pmatrix} = 1+\varepsilon \quad \text{as } t \to 0.$$

It follows from (18) that $\det tM_{4,\dots,m}(t) = \frac{1}{4}+\varepsilon$ as $t \to 0$.

Thus,

$$\lim_{t\to+0}\bigl(tM_{4,\dots,m}(t)\bigr)^{-1} = \begin{pmatrix} \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & 4 \end{pmatrix},$$

and $\zeta_3(0) = \lim_{t\to+0}\zeta_3(t)$. The lemma is proven.

Define $V(p_t)$ at $t=0$ to be the linear span of $\zeta_1(0)$, $\zeta_2(0)$, and $\zeta_3(0)$.

The following assertion is now obvious:

Lemma 6. *The functions $A(\zeta_i(t),\zeta_j(t))$ and $B(p_t;\zeta_i(t),\zeta_j(t))$ are continuous for $t\in[0,1)$ and $i,j\in\{1,2,3\}$.*

Continue the proof of Proposition 4. Given an odd m, put

$$\xi_1 = \frac{1}{2}(-\zeta_1(0)+\zeta_2(0)+\zeta_3(0)),$$

$$\xi_2 = \frac{1}{2}(\zeta_1(0)-\zeta_2(0)+\zeta_3(0)), \quad \xi_3 = \frac{1}{2}(\zeta_1(0)+\zeta_2(0)-\zeta_3(0)).$$

It is easy to verify that the following equalities are valid:

$$(a_{ij}) = (A(\xi_i,\xi_j)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (b_{ij}) = (B(p_0;\xi_i,\xi_j)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Consequently, we obtain $\mu_1 = 1$, $\mu_2 = 0$, and $\mu_3 = -1$ on Γ_0.

Assume that for even m we have

$$\xi_1 = -\frac{\sqrt{39}}{78}(5\zeta_1(0)+5\zeta_2(0)-3\zeta_3(0)), \quad \xi_2 = -\frac{\sqrt{6}}{6}(\zeta_1(0)-2\zeta_2(0)), \quad \xi_3 = \frac{\sqrt{2}}{2}\zeta_1(0).$$

Then (a_{ij}) is the identity matrix and (b_{ij}) has the characteristic polynomial $13\mu^3 - 12\mu^2 - 21\mu + 4$. The zeros of this polynomial are μ_1, μ_2, and μ_3. It is easy to verify that μ_1, μ_2, and μ_3 are pairwise distinct.

Thus, for every m, there is a torus Γ_0 such that the numbers μ_1, μ_2, and μ_3 are pairwise distinct. By Lemma 6, the entries of the Gram–Schmidt matrix (17) are continuous on $[0, T)$. Therefore, the numbers μ_1, μ_2, and μ_3 depend continuously on t in $[0, T)$ as the roots of the characteristic polynomial of the Gram–Schmidt matrix. Thereby there is T' such that the numbers μ_1, μ_2, and μ_3 are pairwise distinct for $t \in [0, T')$. Proposition 4 is proven.

Let

$$a_k = \int\limits_{\gamma_1} \xi_k^2 \frac{dz}{w}, \quad b_k = \int\limits_{\gamma_2} \xi_k^2 \frac{dz}{w}, \quad k = 1, 2, 3. \tag{20}$$

The equalities

$$a_k = \int\limits_{\gamma_1} \xi_k^2 \frac{dz}{w} = -8\eta_1 A(\xi_k, \xi_k) + 8\omega_1 B(\xi_k, \xi_k) = -8\eta_1 + 8\omega_1\mu_k, \quad k = 1, 2, 3,$$

$$b_k = \int\limits_{\gamma_2} \xi_k^2 \frac{dz}{w} = -8\eta_2 A(\xi_k, \xi_k) + 8\omega_2 B(\xi_k, \xi_k) = -8\eta_2 + 8\omega_2\mu_k, \quad k = 1, 2, 3,$$

follow from (5), (16), and (17). Therefore,

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = 8 \begin{pmatrix} -\eta_1 & \omega_1 \\ -\eta_2 & \omega_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix}.$$

It is difficult to estimate the periods a_1, a_2, a_3, b_1, b_2, and b_3 and even more difficult to calculate them. In the following lemma we give some conditions on these periods:

Lemma 7. *Suppose that a torus Γ is such that μ_1, μ_2, and μ_3 are distinct. Then*
(1) *the periods a_1, a_2, and a_3 are reals and the periods b_1, b_2, and b_3 are imaginary;*
(2) *the periods $\xi_k^2 dz/w$ along different cycles do not vanish simultaneously:*

$$|a_k| + |b_k| \neq 0, \quad k = 1, 2, 3;$$

(3) *there is at most one pair (if any) of noncoinciding indices $k, l \in \{1, 2, 3\}$ such that*

$$a_k b_l + a_l b_k = 0;$$

(4) *there is at most one pair (if any) of noncoinciding indices $k, l \in \{1, 2, 3\}$ such that*

$$a_k b_l - a_l b_k = 0;$$

(5) *there is at most one index $k \in \{1, 2, 3\}$ (if any) such that*

$$a_k = 0;$$

this assertion is also valid for $b_k = 0$;
(6) *for different $k, l \in \{1, 2, 3\}$ we have*

$$|a_k b_l - a_l b_k| + |a_k b_l + a_l b_k| \neq 0.$$

Proof. 1. Consider the involution $\tau : (z, w) \mapsto (\bar{z}, \overline{w})$. The following equality is valid for the 1-forms $\varphi = dz/w$ and $z dz/w$:

$$\tau^* \varphi = \overline{\varphi}.$$

Then we have the equalities

$$\int\limits_{\gamma_1} \varphi = \int\limits_{\tau\gamma_1} \varphi = \int\limits_{\tau\tau\gamma_1} \tau^*\varphi = \overline{\int\limits_{\gamma_1} \varphi}.$$

The first equality follows from $\gamma_1 = \tau\gamma_1$. The second equality is valid for arbitrary involutions. The third is obtained from $\tau^*\varphi = \overline{\varphi}$.

Similarly, using $\gamma_2 = -\tau\gamma_2$, we obtain the equality

$$\int\limits_{\gamma_2} \varphi = \int\limits_{-\tau\gamma_2} \varphi = \int\limits_{-\tau\tau\gamma_2} \tau^*\varphi = -\overline{\int\limits_{\gamma_2} \varphi}.$$

Hence, the periods ω_1 and η_1 are real, while ω_2 and η_2 are purely imaginary. The periods a_k and b_k are linear combinations with real coefficients of ω_1, η_1 and ω_2, η_2 respectively. Thus, the first assertion of the lemma is valid.

2. Since $a_k = -\eta_1 + \mu_k\omega_1$ and $b_k = -\eta_2 + \mu_k\omega_2$ by definition, we have

$$\begin{pmatrix} a_k \\ b_k \end{pmatrix} = \begin{pmatrix} -\eta_1 + \mu_k\omega_1 \\ -\eta_2 + \mu_k\omega_2 \end{pmatrix} = \begin{pmatrix} -\eta_1 & \omega_1 \\ -\eta_2 & \omega_2 \end{pmatrix} \begin{pmatrix} 1 \\ \mu_k \end{pmatrix}. \tag{21}$$

Denote the square matrix in the last expression by Δ. It follows from the Legendre equality $\eta_1\omega_2 - \eta_2\omega_1 = \frac{\pi i}{2}$ which is valid on every torus [9] that $\det\Delta = -\eta_1\omega_2 + \eta_2\omega_1 \neq 0$. By the linear independence of the columns of Δ, the vector $(\, a_k \quad b_k \,)$ cannot be zero. Consequently, the second assertion of the lemma is valid.

3. From (21) we obtain the equality

$$a_l b_k + a_k b_l = (\, a_l \quad b_l \,) \begin{pmatrix} b_k \\ a_k \end{pmatrix} = (\, 1 \quad \mu_l \,)\, \Delta^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta \begin{pmatrix} 1 \\ \mu_k \end{pmatrix},$$

where $k, l \in \{1, 2, 3\}$; therefore,

$$\begin{pmatrix} a_k b_l + a_l b_k \\ a_k b_j + a_j b_k \end{pmatrix} = \begin{pmatrix} 1 & \mu_l \\ 1 & \mu_j \end{pmatrix} \Delta^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta \begin{pmatrix} 1 \\ \mu_k \end{pmatrix}.$$

Since $\mu_l \neq \mu_j$ and $\det\Delta \neq 0$, the columns of the product

$$\begin{pmatrix} 1 & \mu_l \\ 1 & \mu_j \end{pmatrix} \Delta^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Delta$$

are linearly independent and the vector $(\, a_k b_l + a_l b_k \quad a_k b_j + a_j b_k \,)$ is nonzero.

We have thus validated the third assertion.

4. The fourth assertion is proven by analogy with the third and follows from the fact that

$$\det \left[\begin{pmatrix} 1 & \mu_l \\ 1 & \mu_j \end{pmatrix} \Delta^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Delta \right] \neq 0.$$

5. To prove the fifth assertion, consider a_k and a_l with different indices $k, l \in \{1, 2, 3\}$. The period ω_1 is different from 0.

From the equality

$$\begin{pmatrix} a_k \\ a_l \end{pmatrix} = \begin{pmatrix} -\eta_1 + \mu_k\omega_1 \\ -\eta_1 + \mu_l\omega_1 \end{pmatrix} = \begin{pmatrix} 1 & \mu_k \\ 1 & \mu_l \end{pmatrix} \begin{pmatrix} -\eta_1 \\ \omega_1 \end{pmatrix}$$

and $\mu_k \neq \mu_l$ we obtain the fifth assertion.

Note that the fifth assertion is also valid for b_k, $k = 1, 2, 3$.

6. The sixth assertion follows from the second and fifth.

The lemma is proven.

Proof of Proposition 3. Henceforth we assume that μ_1, μ_2, and μ_3 are distinct. In Proposition 4 we demonstrated existence of such a torus.

By Lemma 7, renumbering ξ_1, ξ_2, and ξ_3 we can guarantee that $a_3 \neq 0$ and $a_1b_3 + a_3b_1 \neq 0$; therefore, henceforth we assume that $a_3 \neq 0$ and $a_1b_3 + a_3b_1 \neq 0$.

The period problem

$$\operatorname{Re}\int\limits_\gamma \psi_1\psi_2\frac{dz}{w} = 0, \quad \overline{\int\limits_\gamma \psi_1^2\frac{dz}{w}} - \int\limits_\gamma \psi_2^2\frac{dz}{w} = 0, \quad \gamma = \gamma_1, \gamma_2,$$

consists of two real and two complex equations.

Insert ψ_1 and ψ_2 into the first part of the condition:

$$\int\limits_\gamma \psi_1\psi_2\frac{dz}{w} = \int\limits_\gamma \big(vx\xi_1^2 + y\xi_2^2 + (vy + x)\xi_1\xi_2 + vu\xi_1\xi_3 + u\xi_2\xi_3\big)\frac{dz}{w}$$
$$= \int\limits_\gamma \big(vx\xi_1^2 + y\xi_2^2\big)\frac{dz}{w} = \begin{cases} vxa_1 + ya_2 & \text{for } \gamma = \gamma_1, \\ vxb_1 + yb_2 & \text{for } \gamma = \gamma_2. \end{cases}$$

The second equality follows from Lemma 1. The last equality is valid by (20). By the first assertion of Lemma 7, the periods

$$vxa_1 + ya_2 = -(a_1b_2 - a_2b_1)s \in i\mathbb{R}, \quad vxb_1 + yb_2 = (a_1b_2 - a_2b_1)r \in i\mathbb{R}$$

are purely imaginary; therefore, the first part of the period problem is satisfied.

The second part of the condition is simplified by analogy with the first:

$$\overline{\int\limits_\gamma \psi_1^2\frac{dz}{w}} - \int\limits_\gamma \psi_2^2\frac{dz}{w} = \overline{\int\limits_\gamma \big(v^2\xi_1^2 + \xi_2^2\big)\frac{dz}{w}} - \int\limits_\gamma \big(x^2\xi_1^2 + y^2\xi_2^2 + u^2\xi_3^2\big)\frac{dz}{w}$$
$$= \begin{cases} \overline{v^2a_1 + a_2} - x^2a_1 - y^2a_2 - u^2a_3 & \text{for } \gamma = \gamma_1, \\ \overline{v^2b_1 + b_2} - x^2b_1 - y^2b_2 - u^2b_3 & \text{for } \gamma = \gamma_2. \end{cases}$$

Note that, by the first assertion of Lemma 7, the periods a_1, a_2, and a_3 are real. The choice of u in the form (7) guarantees that the periods vanish along the cycle γ_1. Show that the choice of v in the form (6) implies that the periods along γ_2 are zero.

From the form of $v(r, s)$ we conclude that v is a root of the polynomial $cv^4 + (|c|^2 + 2i\operatorname{Im} d)v^2 + \bar{c}d = 0$. Denote $|c|^2 + 2i\operatorname{Im} d$ by α. The discriminant of this polynomial is real:

$$\alpha^2 - 4|c|^2d = |c|^4 + 4|c|^2 i\operatorname{Im} d - 4\operatorname{Im}^2 d - 4|c|^2d = |c|^4 - 4|c|^2\operatorname{Re} d - 4\operatorname{Im}^2 d \in \mathbb{R}.$$

Show that it is positive for (r, s) in some domain $\Omega \subset \mathbb{R}^2$. Note that $d(r, s)$, $c(r, s)$, and the discriminant depend continuously on (r, s) on $\mathbb{R}^2$. Consequently, it suffices to indicate the points in $\mathbb{R}^2$ at which the discriminant is positive.

If $a_2b_3 + a_3b_2 \neq 0$ then the discriminant is positive for $(r, s) = (0, 0)$:

$$\left|\frac{a_2b_3 + a_3b_2}{a_1b_3 + a_3b_1}\right|^2 > 0.$$

If $a_2b_3 + a_3b_2 = 0$ then at $s = 0$ the discriminant equals

$$a_1^4r^6\left|\frac{a_2b_3 - a_3b_2}{a_1b_3 + a_3b_1}\right|^2\left(a_1^4\left(\frac{a_2b_3 - a_3b_2}{a_1b_3 + a_3b_1}\right)^2r^2 + 4a_2^2\frac{a_1b_3 - a_3b_1}{a_1b_3 + a_3b_1}\right)$$

and is nonnegative for a sufficiently large r. Consequently, there is a domain $\Omega \subset \mathbb{R}^2$ where the discriminant is nonnegative.

Now, insert the values of u, x, and y into the expression $\overline{v^2b_1+b_2}-x^2b_1-y^2b_2-u^2b_3$ for the period, multiply by a_3v^2, and divide by $-(a_1b_3+a_3b_1)$. Let $a_3v^2\neq 0$. Obvious calculations demonstrate that the period $\overline{v^2b_1+b_2}-x^2b_1-y^2b_2-u^2b_3$ is zero if and only if $\bar v^2v^2+cv^2+d=0$.

Insert (6) into $\bar v^2v^2+cv^2+d$. Given $(r,s)\in\Omega$, find

$$
\begin{aligned}
\bar v^2v^2+cv^2+d&=\frac{1}{4|c|^2}(\overline{\alpha}\pm\sqrt{\alpha^2-4|c|^2d})(\alpha\pm\sqrt{\alpha^2-4|c|^2d})\alpha^2\\
-\frac12(\alpha\pm\sqrt{\alpha^2-4|c|^2d})+d&=\frac{1}{4|c|^2}(|c|^2\pm\sqrt{\alpha^2-4|c|^2d}-2i\operatorname{Im}d)\\
\times(|c|^2\pm\sqrt{\alpha^2-4|c|^2d}+2i\operatorname{Im}d)\alpha^2&-\frac12(\alpha\pm\sqrt{\alpha^2-4|c|^2d})+d\\
=\frac{1}{4|c|^2}(|c|^4&\pm2|c|^2\sqrt{\alpha^2-4|c|^2d}+\alpha^2-4|c|^2d+4\operatorname{Im}^2d)\\
-\frac12(\alpha\pm\sqrt{\alpha^2-4|c|^2d})+d&=\frac{|c|^2}{4}\pm\frac{\sqrt{\alpha^2-4|c|^2d}}{2}\\
+\frac{\alpha^2+4\operatorname{Im}^2d}{4|c|^2}-d-\frac{|c|^2}{2}&-i\operatorname{Im}d-\frac{\pm\sqrt{\alpha^2-4|c|^2d}}{2}+d=0.
\end{aligned}
$$

Thus, for $(r,s)\in\Omega$ the periods (3) along γ_1 and γ_2 are zero.

Correctness of the construction.

Lemma 8. *The zero sets of $c(r,s)$ and $v(r,s)$ have measure zero in $\mathbb{R}^2$.*

Proof. It follows from the second assertion of Lemma 7 that the polynomial $(a_1r+b_1s)^2$ is nonzero. By the sixth assertion of Lemma 7, the function $c(r,s)$ is not identically zero. Hence, the zero set of $c(r,s)$ has measure zero.

The roots of the polynomial $cv^4+(|c|^2+2i\operatorname{Im}d)v^2+\bar cd$ are zero if and only if $|c|^2+2i\operatorname{Im}d=0$ and $\bar cd=0$. Consequently, a necessary condition for $v(r,s)=0$ is $c(r,s)=0$. The lemma is proven.

Thus, the expression which defines ψ_1 and ψ_2 contains no division by zero for almost all $(r,s)\in\Omega$.

Proposition 5. *There exist $(r,s)\in\Omega$ such that ψ_1 and ψ_2 have poles at each of the points $P_0,\dots,P_3,P_4^{\pm},\dots,P_m^{\pm}$.*

Denote the submatrices composed of the columns $j_1,\dots,j_m$ of the matrix M by $M_{j_1,\dots,j_m}$.

Lemma 9. *For almost all points $p_1,\dots,p_m\in\mathbb{R}$ such that $p_1+p_2+p_3=0$ all square submatrices $M_{j_1,\dots,j_m}$ are nondegenerate.*

Proof. Interchanging the variables p_i and p_j for $3\le i,j\le m$ $(1\le i,j\le 3)$ results in interchanging the ith and jth columns and the $(i-3)$th and $(j-3)$th rows of M. Therefore, it suffices to prove nondegeneracy of $M_{4,\dots,m}$, $M_{1,4,\dots,m-2}$, $M_{1,2,4,\dots,m-1}$, and $M_{1,\dots,m-3}$.

Nondegeneracy of $M_{4,\dots,m}$ is proven in Lemma 2.

Let N be a sufficiently large natural number. Put

$$p=(p_1,\dots,p_m)=(-1,0,1,N,N+1,2N,2N+1,\dots).$$

The submatrices $M_{1,4,\dots,m-1}$, $M_{1,2,4,\dots,m-2}$, and $M_{1,\dots,m-3}$ have the form

$$\begin{pmatrix} R & S\\ T & U\end{pmatrix},$$

where S stands for $M_{4,\dots,m-1}$, $M_{4,\dots,m-2}$, or $M_{4,\dots,m-3}$. By Lemma 2, the submatrix S is nondegenerate in each case. The entries of R have the asymptotic behavior $\varepsilon=O(1/N)$ as $N\to\infty$. The entries of U are bounded by a constant independent of N.

The submatrix T has one of the following forms:

$$\left(\frac{1}{p_m-p_1}\right), \begin{pmatrix} \frac{1}{p_{m-1}-p_1} & \frac{1}{p_{m-1}-p_2} \\ \frac{1}{p_m-p_1} & \frac{1}{p_m-p_2} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \frac{1}{p_{m-2}-p_1} & \frac{1}{p_{m-2}-p_2} & \frac{1}{p_{m-2}-p_3} \\ \frac{1}{p_{m-1}-p_1} & \frac{1}{p_{m-1}-p_2} & \frac{1}{p_{m-1}-p_3} \\ \frac{1}{p_m-p_1} & \frac{1}{p_m-p_2} & \frac{1}{p_m-p_3} \end{pmatrix}.$$

Choose the numbers p_{m-2}, p_{m-1}, $p_m \in (0,1)$ (only those entering T) so that the determinant T be nonzero. Then the matrix

$$\begin{pmatrix} R & S \\ T & U \end{pmatrix} \to \begin{pmatrix} \varepsilon & S \\ T & * \end{pmatrix}$$

splits into two matrices with nondegenerate determinants as $N \to \infty$.

Estimate $M_{1,4,\dots,m-1}$, $M_{1,2,4,\dots,m-2}$, and $M_{1,\dots,m-3}$ to find:

$$\det \begin{pmatrix} \varepsilon & S \\ T & * \end{pmatrix} = \det S \det T + O(1/N).$$

The lemma is proven.

Prove the following consequence of Lemma 9:

Lemma 10. *For almost all $p = (p_1, \dots, p_m)$ such that $p_1 + p_2 + p_3 = 0$, and for each of the points $P_0, \dots, P_3, P_4^{\pm}, \dots, P_m^{\pm}$, there is a function $\psi \in V(p)$ having a pole at this point.*

Proof. By Lemma 2, the space

$$V(p) = \left\{ \sum_{i=1}^{m} \frac{\nu_i w}{z - p_i} \mid (\nu_1, \dots, \nu_k, \dots, \nu_m) \in \ker M \right\}$$

is three-dimensional for almost all $p = (p_1, \dots, p_m)$ such that $p_1 + p_2 + p_3 = 0$.

Suppose the converse; i.e., suppose that the functions in $V(p)$ have no pole at one of the points $(p_k, w_k) \in \{P_1, P_2, P_3, P_4^{\pm}, \dots, P_m^{\pm}\}$, $k \in \{1, \dots, m\}$. Then

$$\begin{aligned} V(p) &= \left\{ \sum_{i=1}^{m} \frac{\nu_i w}{z - p_i} \mid (\nu_1, \dots, \nu_{k-1}, \nu_k, \nu_{k+1}, \dots, \nu_m) \in \ker M,\ \nu_k = 0 \right\} \\ &= \left\{ \sum_{i=1, i\neq k}^{m} \frac{\nu_i w}{z - p_i} \mid (\nu_1, \dots, \nu_{k-1}, \nu_{k+1}, \dots, \nu_m) \in \ker M' \right\}, \end{aligned}$$

where M' is the matrix M with the kth column deleted.

By Lemma 9, the matrix M' has the maximal rank for almost all points $p_1, \dots, p_m$. Consequently, $\dim_{\mathbb{C}} \ker M' = m - 1 - \operatorname{rank} M' = 2$, while $\dim_{\mathbb{C}} V(p) = \dim_{\mathbb{C}} \ker M = \dim_{\mathbb{C}} \ker M' = 3$; a contradiction.

Now, we are left with validating the assertion for the point P_0.

Suppose the converse. Assume that the functions in $V(p)$ have no pole at P_0:

$$\begin{aligned} V(p) &= \left\{ \sum_{i=1}^{m} \frac{\nu_i w}{z - p_i} \mid (\nu_1, \dots, \nu_m) \in \ker M,\ \nu_1 + \dots + \nu_m = 0 \right\} \\ &= \left\{ \sum_{i=1}^{m} \frac{\nu_i w}{z - p_i} \mid (\nu_1, \dots, \nu_m) \in \ker M'' \right\}, \end{aligned}$$

where M'' is the matrix M augmented by a row of unities.

By analogy with the proof of Lemma 9, we can easily prove that the matrix M'' has the maximal rank for almost all points $p_1, \dots, p_m$. Consequently, $\dim_{\mathbb{C}} \ker M'' = m - \operatorname{rank} M'' = 2$, while $\dim_{\mathbb{C}} V(p) = \dim_{\mathbb{C}} \ker M = \dim_{\mathbb{C}} \ker M'' = 3$, a contradiction. The lemma is proven.

Lemma 11. *The function $v(r,s)$ is nonconstant on Ω.*

Proof. By Lemma 8, the polynomial $c(r,s)$ is nonzero for almost all $(r,s) \in \mathbb{R}^2$. For $c(r,s) \neq 0$ the function v is a root of the polynomial

$$v^2 + \frac{(|c|^2 + 2i\,\mathrm{Im}\, d)}{c} v + \frac{\bar{c}d}{c} = 0. \tag{22}$$

Suppose the converse; i.e., suppose that the zeros of (22) are constant. This means that $(|c|^2 + 2i\,\mathrm{Im}\, d)/c$ and $\bar{c}d/c$ are constant. It follows from here and the form of the functions $c(r,s)$ and $d(r,s)$ that $d(r,s)$ is zero and $c(r,s)$ is constant.

Consequently, $a_1b_3 - a_3b_1$ and $a_2b_3 - a_3b_2$ vanish simultaneously which contradicts the fourth assertion of Lemma 7. The lemma is proven.

Lemma 12. *The zero set of $u(r,s)$ has measure zero in $\mathbb{R}^2$.*

Proof. If $s = 0$ and $r \to \infty$ then we obtain

$$v^2(r,s) = -\frac{a_2b_3 - a_3b_2}{a_1b_3 + a_3b_1} a_1^2 r^2 + o(r^2), \quad u^2(r,s) = \frac{a_1b_2 + a_2b_1}{a_1b_3 + a_3b_1} a_1^2 r^2 + o(r^2). \tag{23}$$

If $a_1 \neq 0$ and $a_1b_2 + a_2b_1 \neq 0$ then (23) implies that $u(s,t) \neq 0$ for almost all $(r,s) \in \mathbb{R}^2$. If $a_1 = 0$ then the assertion of the lemma is obvious.

If $a_1b_2 + a_2b_1 = 0$ then it is easy to validate the equalities

$$v^2(0,0) = -\frac{a_2b_3 + a_3b_2}{a_1b_3 + a_3b_1}, \quad u^2(0,0) = -\frac{a_1}{a_3}\frac{a_1b_2 - a_2b_1}{a_1b_3 + a_3b_1}.$$

By the sixth assertion of Lemma 7, from $a_1b_2 + a_2b_1 = 0$ we obtain $a_1b_2 - a_2b_1 \neq 0$.

Thus, $u(r,s)$ is a nonzero algebraic function. Hence the assertion of the lemma follows. The lemma is proven.

Proof of Proposition 5. Assume that ψ_1 has no pole at Q for all $(r,s) \in \Omega$. Then ξ_1 and ξ_2 have no pole at Q. This follows from the form of ψ_1 and the fact that $v(r,s)$ changes on Ω by Lemma 11.

Therefore, by Lemma 10, ξ_3 has a pole at Q. Thereby, $\psi_2 = x(r,s)\xi_1 + y(r,s)\xi_2 + u(r,s)\xi_3$ has a pole at Q for r and s such that $u(r,s) \neq 0$; i.e., ψ_2 has a pole for almost all $(r,s) \in \mathbb{R}^2$.

Thus, at each of the points $P_0, \dots, P_3, P_4^{\pm}, \dots, P_m^{\pm}$ at least one of the functions ψ_1 and ψ_2 has a simple pole. So, the surface (2) has exactly $n = 2m - 2 \geq 6$ planar embedded ends. The proposition is proven.

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